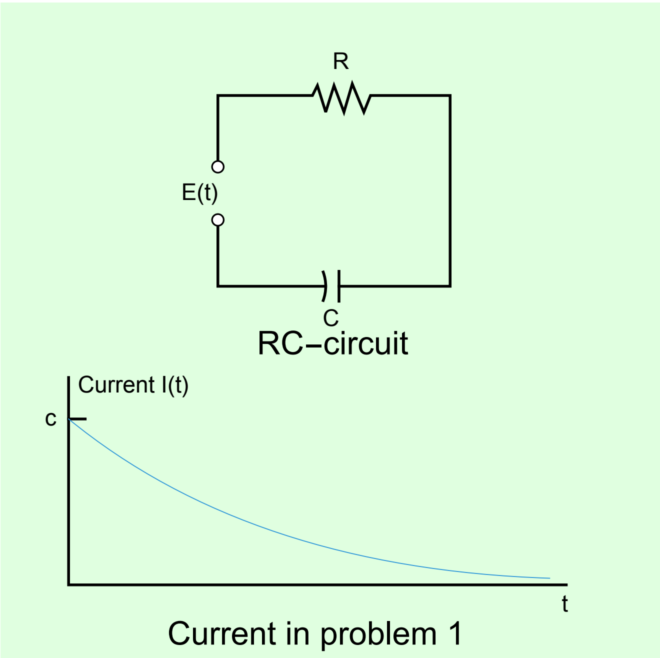
1 - 6 RLC-Circuits: special cases

1. RC-Circuit. Model the RC-Circuit in the figure below. Find the current due to a constant E.




```
ClearAll["Global`*"]
```

The problem is asking for a look at RC circuit, not RLC.

The site *https : // www.intmath.com/differential - equations/6 - rc - circuits.php* assumes a constant voltage source, just what the problem specifies. Below: There is no inductance here, only R and C.

```
eqnw = rR (D[eye[t], t]) + eye[t] / cC == 0
```

$$\frac{\text{eye}[t]}{\text{cC}} + \text{rR}\,\text{eye}'[t] == 0$$

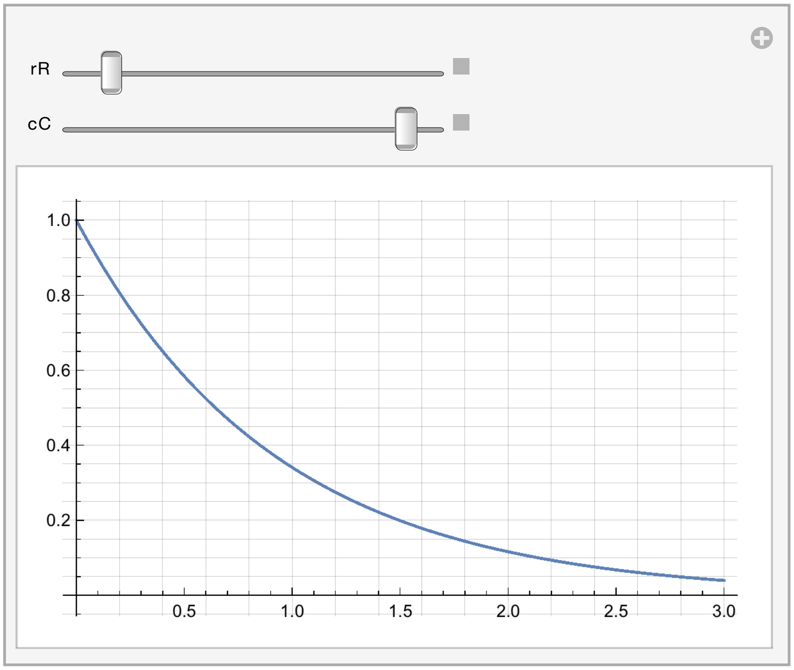
Within a certain range of capacitance and resistance, the plot resembles the one in the problem description, and can be manipulated to imitate changing parameters, with the voltage remaining constant.

```
sol2 = DSolve[eqnw, eye, t]
```

$$\left\{\left\{\text{eye} \to \text{Function}\left[\{t\},\ e^{-\frac{t}{\text{cC rR}}}\, C[1]\right]\right\}\right\}$$

It looks like the current is normalized to 1 at t=0, and the fraction of its max value at a given time needs to be estimated from the underlying grid.

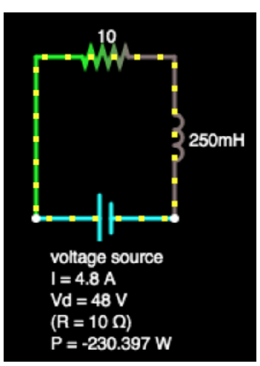
```
Manipulate[Plot[ⅇ^(-t/(cC rR)), {t, 0, 3}, PlotRange → All, GridLines → All
 ], {rR, 0.2, 10}, {cC, 0.01, 1} ]
```



A random scrap from a different perspective, kept as interesing junk.

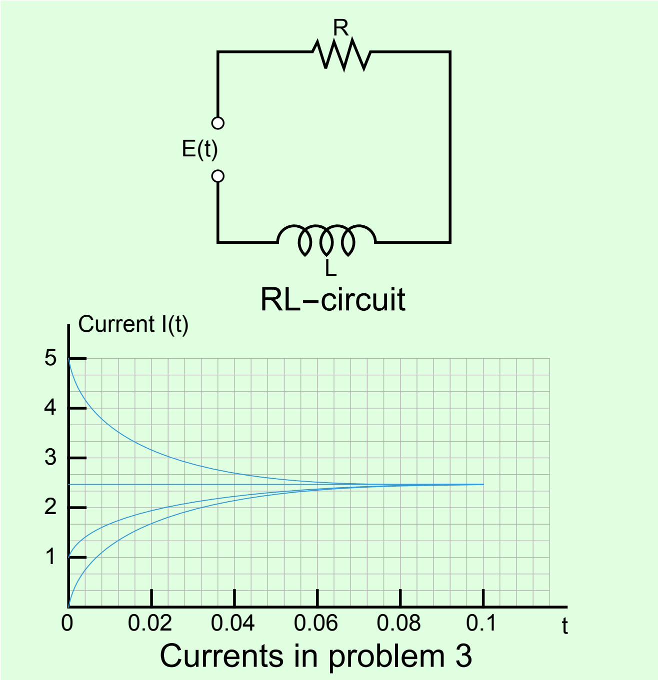
```
{ind, cap, res} = {l i'[t] == v1[t], vc'[t] == 1 / c i[t], r i[t] == vr[t]};
kirchhoff = v1[t] + vc[t] + vr[t] == vs[t];
```

3. RL-Circuit. Model the RL-circuit in the figure below. Find a general solution when R, L, E are any constants. Graph or sketch solutions when L = 0.25 H, R = 10 Ω, and E = 48 V.



The above screenshot came from the online app at *https://falstad.com/circuit/*. The current it shows agrees with the old formula for current, I=E/R, and was captured after the resistance had plenty of time to decay. And that’s all it is, except that there is a time constant to apply. The time constant becomes ever smaller as the operation time increases. Since the problem description talks in terms of a constant state, it seems the time constant would become vanishingly small, leaving merely I=E/R=4.8 amps.

In[60]:=
```
ClearAll["Global`*"]
```



When there are a lot of variables to watch, the Manipulate command is the only way I know to get an overview. The box below is based on the material at *https://www.electronics-tutorials.ws/inductor/lr-circuits.html* and may not agree with the text in detail.

In[61]:= eye[vee_, are_, ell_, tee_] = $\frac{\text{vee}}{\text{are}}\left(1 - e^{-\frac{\text{are tee}}{\text{ell}}}\right)$

Out[61]= $\frac{\left(1 - e^{-\frac{\text{are tee}}{\text{ell}}}\right)\text{vee}}{\text{are}}$

It takes some time for the current to reach its max value. From t=0.4 on in the green grid below, the circuit current is nominal.

In[62]:=
```
Grid[Table[{tee, eye[48, 10, 0.25, tee]}, {tee, 0, 0.6, 0.1}], Frame → All]
```

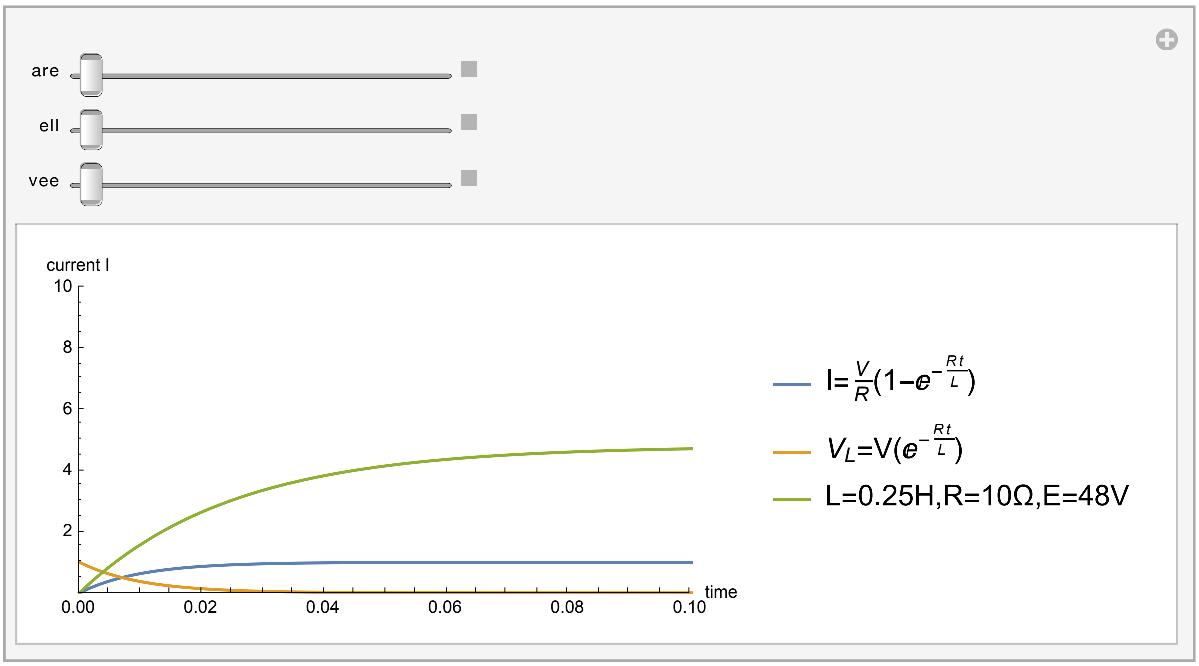
Out[62]=

0.	0.
0.1	4.71208
0.2	4.79839
0.3	4.79997
0.4	4.8
0.5	4.8
0.6	4.8

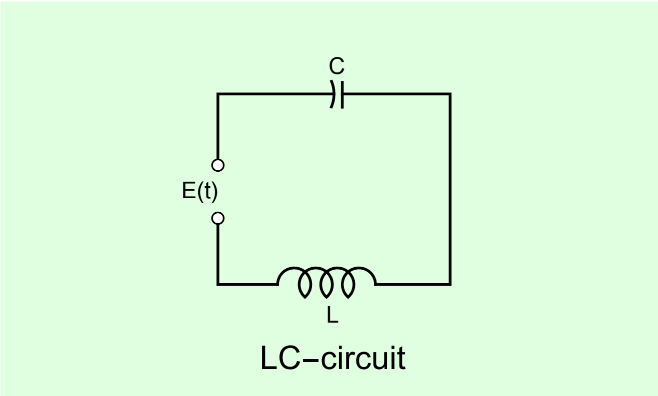
In[63]:=
```
veel[vee_, are_, ell_, tee_] = vee (ⅇ^(-are tee/ell))
```
Out[63]= $e^{-\frac{\text{are tee}}{\text{ell}}}\ \text{vee}$

In[64]:=
```
Manipulate[
 Plot[{Abs[eye[vee, are, ell, tee]], Abs[veel[vee, are, ell, tee]],
   Abs[eye[48, 10, 0.25, tee]]}, {tee, 0, 5},
  PlotLegends → {"I=V/R(1-ⅇ^(-R t/L))", "V_L=V(ⅇ^(-R t/L))", "L=0.25H,R=10Ω,E=48V"},
  PlotRange → {{0, 0.1}, {0, 10}}, AxesLabel → {"time", "current I"},
  AspectRatio → 0.5], {are, 1, 200}, {ell, 0.01, 10} , {vee, 1, 50}]
```

Out[64]=



5. LC-Circuit. This is an RLC-circuit with negligibly small R (analog of an undamped mass-spring system). Find the current when L=0.5 H, C = 0.005 F, and E = Sin[t] V, assuming zero initial current and charge.



I ran across a couple of snippets, including one from the *Mathematica* documentation, suggesting that state space modeling would be a good way to look at circuits in Mathematica. I use it here.

```
ClearAll["Global`*"]
```

$$\text{eqns} = \left\{\text{eL}\, q''[t] + \text{aR}\, q'[t] + \frac{1}{\text{cC}} q[t] == \text{Vee}[t]\right\};$$

```
m1 = StateSpaceModel[eqns,
  {{q[t], 0}, {q'[t], 0}}, {{Vee[t], 0}}, {q'[t]}, t]
```

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{\text{cC eL}} & -\frac{\text{aR}}{\text{eL}} & \frac{1}{\text{eL}} \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

Here I put in the given parameters, taking the opportunity to equate the resistance with zero.

```
ms = m1 /. {cC → 0.005, eL → 0.5, aR → 0}
```

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -400. & 0. & 2. \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

The way to get output from a state space model is to use the command **`OutputResponse`**. Since the voltage depends on a periodic function, I drop the V for the input field, the voltage, because it is just a label.

```
outz = OutputResponse[{ms}, Sin[t ], t]
```

$\{(1.46082\times10^{-17} + 0.0526316\,i)$
$((0. + 0.0952381\,i)\,Cos[20.\,t] - (0. + 1.\,i)\,Cos[19.\,t]\,Cos[20.\,t] +$
$(0. + 0.904762\,i)\,Cos[20.\,t]\,Cos[21.\,t] -$
$(1.66533\times10^{-16} - 7.21645\times10^{-17}\,i)\,Cos[20.\,t]\,Sin[19.\,t] -$
$(2.24688\times10^{-17} + 6.60847\times10^{-19}\,i)\,Sin[20.\,t] +$
$(2.35922\times10^{-16} + 6.93889\times10^{-18}\,i)\,Cos[19.\,t]\,Sin[20.\,t] -$
$(2.13454\times10^{-16} + 6.27805\times10^{-18}\,i)\,Cos[21.\,t]\,Sin[20.\,t] +$
$(5.96745\times10^{-17} - 1.\,i)\,Sin[19.\,t]\,Sin[20.\,t] +$
$(1.50673\times10^{-16} - 6.52917\times10^{-17}\,i)\,Cos[20.\,t]\,Sin[21.\,t] -$
$(5.39912\times10^{-17} - 0.904762\,i)\,Sin[20.\,t]\,Sin[21.\,t])\}$

It is necessary to clean up the result with a small **Chop**.

$outt = Chop[ComplexExpand[Re[outz]], 10^{-16}]$ // FullSimplify

$\{0.00501253\,Cos[1.\,t] - 0.00501253\,Cos[20.\,t] + 3.46945\times10^{-18}\,Cos[39.\,t]\}$

Recognizing the periodic value of cosine, I can get the expression ready for a second chop by doing

```
outtf = outt /. Cos[39. t] → 1
```

$\{3.46945\times10^{-18} + 0.00501253\,Cos[1.\,t] - 0.00501253\,Cos[20.\,t]\}$

And then the **Chop**.

$outtff = Chop[\%, 10^{-17}]$

```
{0.00501253 Cos[1. t] - 0.00501253 Cos[20. t]}
```

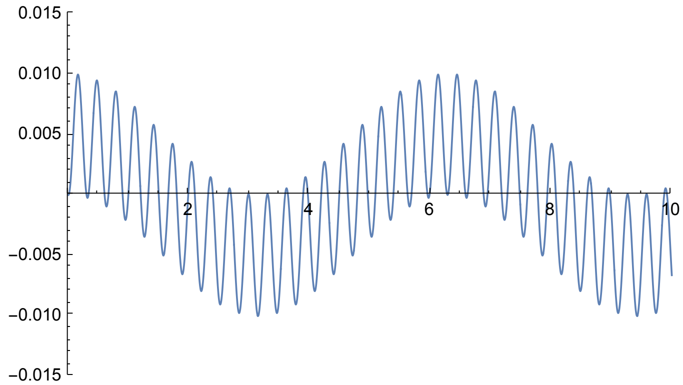
Testing the identity of those coefficients

```
1/0.005012531328320802`
199.5
```

I find that the answer matches the text answer, justifying the green coloration above.

The plot is interesting.

```
Plot[outtff, {t, 0, 10}, ImageSize → 350, AspectRatio → 0.6,
 PlotRange → {{-0.01, 10}, {-0.015, 0.015}}, PlotStyle → Thickness[0.003]]
```



7 - 18 General RLC-circuits

7. Tuning. In tuning a sterio system to a radio station, we adjust the tuning control (turn a knob) that changes C (or perhaps L) in an RLC-circuit so that the amplitude of the steady-state current, numbered line (5), p. 95 becomes maximum. For what C will this happen?

It is where the particular solution of the homogeneous equation is maximized. Numbered line (5) looks like

$I_p(t) = I_0 \sin[\omega t - \Theta]$

The quantity θ is known as the phase lag, and, I suppose, the signal is best, I_p maximized, when θ equals zero.

8 - 14 Find the steady-state current in the RLC-circuit in the figure below for the given data.

9. R = 4 Ω, L = 0.1 H, C = 0.05 F, E = 110 V

$L\, D[q[t], \{t, 2\}] + R\, D[q[t], t] - \frac{1}{C} q[t] = v[t]$

$$\text{eqn} = 0.1\, q''[t] + 4\, q'[t] - \frac{1}{0.05} q[t] == 110$$

$$-20.\, q[t] + 4\, q'[t] + 0.1\, q''[t] == 110$$

```
sol = DSolve[eqn, q, t]
```

$\{\{q \to \text{Function}[\{t\}, -5.5 + e^{-44.4949\, t}\, C[1] + e^{4.4949\, t}\, C[2]]\}\}$

If C[1]=C[2]=0, then the green cell above matches the text answer.

11. R = 12 Ω, L = 0.4 H, C = $\frac{1}{80}$F, E = 220 Sin[10 t] V

The state space method has been working where former methods I tried did not, so it makes sense to stick with it.

```
ClearAll["Global`*"]
```

```
eqns = {eL q''[t] + aR q'[t] + 1/cC q[t] == Vee[t]};
```

```
m1 = StateSpaceModel[eqns,
  {{q[t], 0}, {q'[t], 0}}, {{Vee[t], 0}}, {q'[t]}, t]
```

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{cC\ eL} & -\frac{aR}{eL} & \frac{1}{eL} \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

Here I put in the given parameters.

```
ms = m1 /. {cC → 1/80, eL → 0.4, aR → 12}
```

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -200. & -30. & 2.5 \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

The way to get output from a state space model is to use the command **OutputResponse**.

```
outz = OutputResponse[{ms}, 220 Sin[10 t ], t]
```

$\{0. + e^{-30. t} (22. e^{10. t} - 27.5 e^{20. t} - 7.10543\times 10^{-15} e^{20. t} \text{Cos}[10. t] + 5.5 e^{30. t} \text{Cos}[10. t] + 7.10543\times 10^{-15} e^{20. t} \text{Sin}[10. t] + 16.5 e^{30. t} \text{Sin}[10. t] + 7.10543\times 10^{-15} e^{40. t} \text{Sin}[10. t])\}$

It is necessary to clean up the result with a **Chop**.

```
outt = Chop[outz, 10^-14] // FullSimplify
```

$\{22. e^{-20. t} - 27.5 e^{-10. t} + 5.5 \text{Cos}[10. t] + 16.5 \text{Sin}[10. t]\}$

I guess the *e* factors can be dropped if they are small enough, say, at 3 seconds.

```
N[-27.500000000000007` e^(-10.` t)] /. t → 3
```

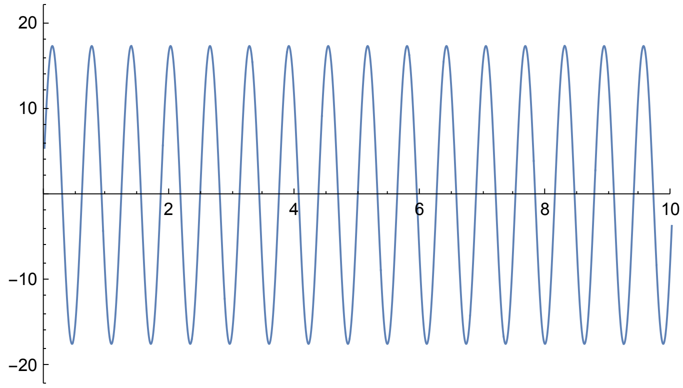
-2.57335×10^{-12}

Evidently the text considers that size to be negligible, leaving

```
5.5 Cos[10. t] + 16.5 Sin[10. t]
```

as the answer. The plot looks routine.

```
Plot[5.5 Cos[10. t] + 16.5 Sin[10. t],
 {t, 0, 10}, ImageSize → 350, AspectRatio → 0.6,
 PlotRange → {{-0.01, 10}, {-22, 22}}, PlotStyle → Thickness[0.003]]
```



13. R = 12, L = 1.2 H, C = $\frac{20}{3}*10^{-3}$ F, E = 12,000 Sin[25 t] V

$C = \frac{20}{3} * \frac{1}{1000} = \frac{20}{3000} = \frac{2}{300}$

```
ClearAll["Global`*"]
```

$$\text{eqns} = \left\{\text{eL}\, q''[t] + \text{aR}\, q'[t] + \frac{1}{\text{cC}} q[t] == \text{Vee}[t]\right\};$$

```
m1 = StateSpaceModel[eqns,
  {{q[t], 0}, {q'[t], 0}}, {{Vee[t], 0}}, {q'[t]}, t]
```

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{\text{cC eL}} & -\frac{\text{aR}}{\text{eL}} & \frac{1}{\text{eL}} \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

Here I put in the given parameters.

$$\text{ms} = \text{m1} \,/.\, \left\{\text{cC} \to \frac{20}{3} * 10^{-3},\ \text{eL} \to 1.2,\ \text{aR} \to 12\right\}$$

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -125. & -10. & 0.833333 \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

The way to get output from a state space model is to use the command **OutputResponse**.

```
outz = OutputResponse[{ms}, 12 000 Sin[25 t ], t]
```

$$\{(0. + 0. i) - (400. + 1.56319 \times 10^{-14} i) e^{-5. t} ((-1. + 0. i) Cos[10. t] + (1. + 0. i) e^{5. t} Cos[10. t]^2 Cos[25. t] + (0.75 - 4.80505 \times 10^{-16} i) Sin[10. t] - (3.19744 \times 10^{-16} - 3.21521 \times 10^{-16} i) e^{5. t} Cos[10. t] Cos[25. t] Sin[10. t] + (1. + 2.45581 \times 10^{-16} i) e^{5. t} Cos[25. t] Sin[10. t]^2 - (0.5 - 7.49623 \times 10^{-17} i) e^{5. t} Cos[10. t]^2 Sin[25. t] + (1.42109 \times 10^{-16} - 9.97247 \times 10^{-17} i) e^{5. t} Cos[10. t] Sin[10. t] Sin[25. t] - (0.5 - 1.39035 \times 10^{-16} i) e^{5. t} Sin[10. t]^2 Sin[25. t])\}$$

It is necessary to clean up the result with a **Chop**.

```
outt = Chop[ComplexExpand[Re[outz]], 10^-15] // Simplify
```

$$\{-300. e^{-5. t} Sin[10. t] + Cos[10. t] (400. e^{-5. t} + 1.27898 \times 10^{-13} Cos[25. t] Sin[10. t]) - 2.84217 \times 10^{-14} Sin[20. t] Sin[25. t] + Cos[10. t]^2 (-400. Cos[25. t] + 200. Sin[25. t]) + Sin[10. t]^2 (-400. Cos[25. t] + 200. Sin[25. t])\}$$

There is a $sin^2 + cos^2$ trig identity in the above, but I'm going to have to pull it out by hand.

```
outhnd = -300. e^(-5. t) Sin[10. t] +
  Cos[10. t] (400. e^(-5. t)) + (-400. Cos[25. t] + 200. Sin[25. t])
```

$$400. e^{-5. t} Cos[10. t] - 400. Cos[25. t] - 300. e^{-5. t} Sin[10. t] + 200. Sin[25. t]$$

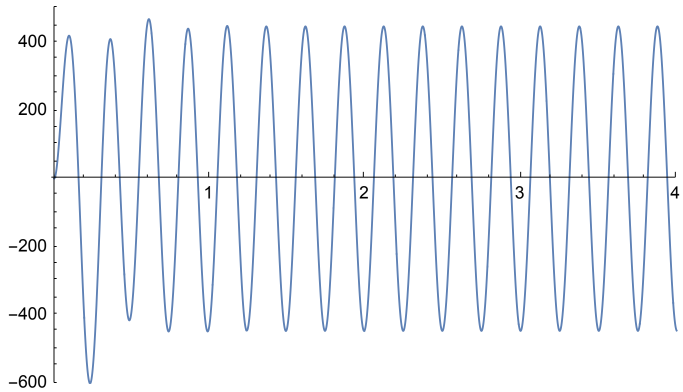
```
outhnd2 = Collect[outhnd, e^(-5. t)]

Clear["Global`*"]
```

$$-400. Cos[25. t] + e^{-5. t} (400. Cos[10. t] - 300. Sin[10. t]) + 200. Sin[25. t]$$

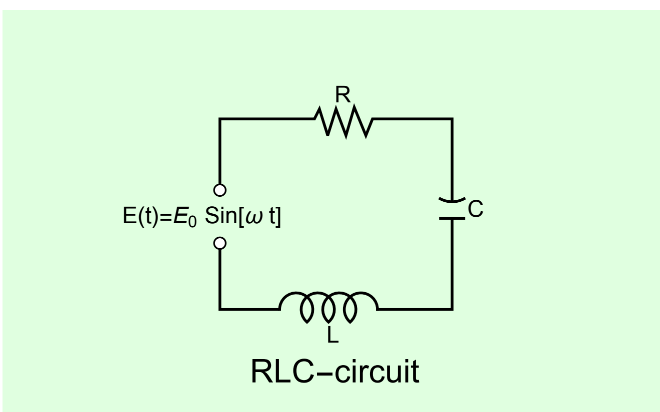
While I was pulling things out by hand, I pulled out a choppable term. The text constant B is equal to -300. The text constant A is equal to 1 in one position and 400 in another position. That makes my answer wrong, technically. I guess I should make it yellow, though I don't feel it is a just action to do so. I feel like it is correct.

```
Plot[-400.` Cos[25.` t] + ⅇ^(-5.` t) (400.` Cos[10.` t] - 300.` Sin[10.` t]) +
  200.` Sin[25.` t], {t, 0, 4}, ImageSize → 350, AspectRatio → 0.6,
 PlotRange → {{-0.01, 4}, {-600, 500}}, PlotStyle → Thickness[0.003]]
```



15. Cases of damping. What are the conditions for an RLC-circuit to be (I) overdamped, (II) critically damped, (III) underdamped? What is the critical resistance R_{crit} (the analog of the critical damping constant $2\sqrt{m\,k}$?

16 - 18 Solve the initial value problem for the RLC-circuit shown below, with the given data, assuming zero initial current and charge. Graph or sketch the solution.



17. R = 6 Ω, L = 1 H, C = 0.04 F, E = 600(Cos[t] + 4 Sin[t])V

```
ClearAll["Global`*"]
```

$$\texttt{eqns} = \left\{\texttt{eL q''[t] + aR q'[t]} + \frac{1}{\texttt{cC}}\texttt{q[t]} == \texttt{Vee[t]}\right\};$$

```
m1 = StateSpaceModel[eqns,
  {{q[t], 0}, {q'[t], 0}}, {{Vee[t], 0}}, {q'[t]}, t]
```

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{cC\,eL} & -\frac{aR}{eL} & \frac{1}{eL} \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

Here I put in the given parameters.

```
ms = m1 /. {cC → 0.04, eL → 1, aR → 6}
```

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -25. & -6 & 1 \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

The way to get output from a state space model is to use the command **OutputResponse**.

```
outz = OutputResponse[{ms}, 600 (Cos[t] + 4 Sin[t]), t]
```

$$\{(0. + 0.\,i) + e^{-3.\,t}\,((-100. - 1.11022\times 10^{-14}\,i)\,Cos[4.\,t] + (100. + 1.11022\times 10^{-14}\,i)\,e^{3.\,t}\,Cos[t]\,Cos[4.\,t]^2 - (1.87214\times 10^{-14} - 1.65445\times 10^{-14}\,i)\,e^{3.\,t}\,Cos[4.\,t]^2\,Sin[t] + (75. + 1.52656\times 10^{-14}\,i)\,Sin[4.\,t] - (8.65974\times 10^{-15} + 1.80411\times 10^{-14}\,i)\,e^{3.\,t}\,Cos[t]\,Cos[4.\,t]\,Sin[4.\,t] + (2.27374\times 10^{-13} + 2.91161\times 10^{-14}\,i)\,e^{3.\,t}\,Cos[4.\,t]\,Sin[t]\,Sin[4.\,t] + (100. - 1.14492\times 10^{-14}\,i)\,e^{3.\,t}\,Cos[t]\,Sin[4.\,t]^2 - (0. + 7.91555\times 10^{-14}\,i)\,e^{3.\,t}\,Sin[t]\,Sin[4.\,t]^2)\}$$

```
outt = Chop[ComplexExpand[Re[outz]]] // Simplify
```

$$\{-100.\,e^{-3.\,t}\,Cos[4.\,t] + 100.\,Cos[t]\,Cos[4.\,t]^2 + Sin[4.\,t]\,(75.\,e^{-3.\,t} + 100.\,Cos[t]\,Sin[4.\,t])\}$$

$$outtf = Collect[outt, e^{-3.\,t}]$$

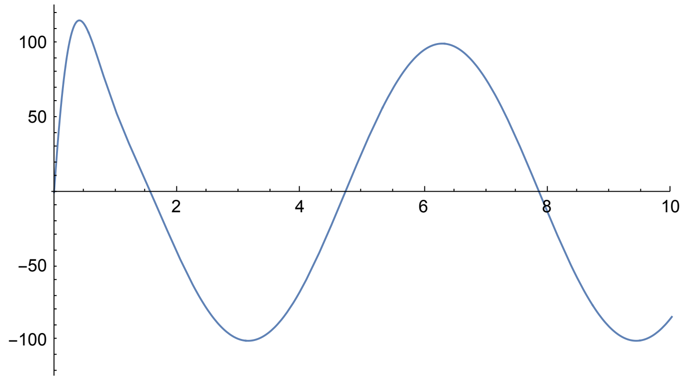
$$\{100.\,Cos[t]\,Cos[4.\,t]^2 + 100.\,Cos[t]\,Sin[4.\,t]^2 + e^{-3.\,t}\,(-100.\,Cos[4.\,t] + 75.\,Sin[4.\,t])\}$$

I can see the $\sin^2 + \cos^2$ identity in the above, but will have to take it out by hand.

$$100.\,Cos[t] + e^{-3.\,t}\,(-100.\,Cos[4.\,t] + 75.\,Sin[4.\,t])$$

And with that, the above cell matches the text answer.

```
Plot[100. Cos[t] + e^(-3. t) (-100. Cos[4. t] + 75. Sin[4. t]),
 {t, 0, 10}, ImageSize → 350, AspectRatio → 0.6,
 PlotRange → {{-0.01, 10}, {-125, 125}}, PlotStyle → Thickness[0.003]]
```



19. Writing report. Mechanical-electrical analogy. Explain table 2.2 (reproduced below) in a 1 - 2 page report with examples, e.g. the analog (with L = 1 H) of a mass-spring system of mass 5 kg, damping constant 10 kg/sec, spring constant 60 kg/sec^2, and driving force 220 cos 10t kg/sec.

Electrical System	Mechanical System
Inductance L	Mass m
Reciprocal $\frac{1}{c}$ of capacitance	Spring modulus k
Derivative $E_0\omega$ Cos[ω t] of electromotive force	Driving force F_0Cos[ω t]
Current I(t)	Displacement y(t)

The equivalent equations of state are given on p. 97 as

$$L_e * I_e''[t] + R_e * I_e'[t] + \frac{1}{C_e} * I_e[t] = E_0 * \omega * \text{Cos}[\omega t]$$

for the electrical version and

$$m * y''[t] + c * y'[t] + k * y[t] = F_0 \text{Cos}[\omega t]$$

for the mechanical version. The problem details of the mechanical system are set forth as

```
In[28]:= m = 5;
c = 10;
k = 60;
F0 = 220 Cos[10 t];
```

To see if I have the mechanical side down, let me try to get a function for the displacement y.

```
In[5]:= eqn1 = 5 y''[t] + 10 y'[t] + 60 y[t] == 220 Cos[10 t]
Out[5]= 60 y[t] + 10 y′[t] + 5 y″[t] == 220 Cos[10 t]
```

Mathematica solves the equation without difficulty; however, the solution is not as simple an expression as I could wish.

```
In[8]:= sol = DSolve[eqn1, y, t];
```

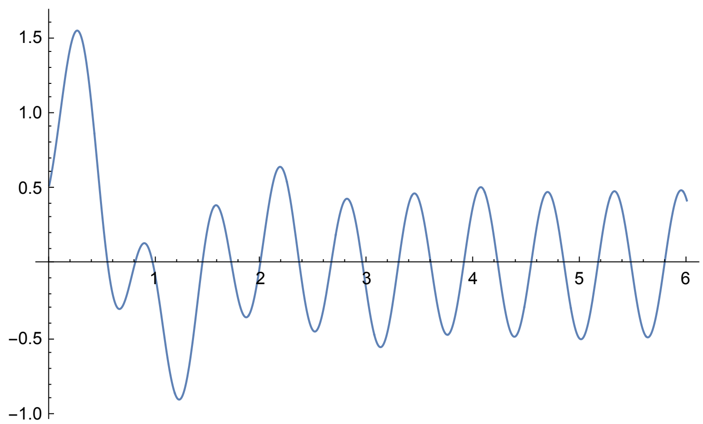
The solution does backtest successfully.

```
In[9]:= eqn1 /. sol // Simplify
Out[9]= {True}
```

And the resulting plot is typical of a forced SHM.

```
In[51]:= solp = sol /. {C[1] → 1, C[2] → 1};
In[23]:= Plot[y[t] /. solp, {t, 0, 6}, PlotStyle → Thickness[0.003]]
```

Out[23]=



I can extract the actual function

```
In[47]:= solpt = solp[[1, 1, 2, 2]];
```

and make a table of a few of its output points.

```
In[53]:= Table[{n, solpt /. t → n}, {n, 0, 2, 0.1}]
Out[53]= {{0., 0.524558}, {0.1, 0.984198}, {0.2, 1.44535}, {0.3, 1.51068},
  {0.4, 1.04147}, {0.5, 0.312714}, {0.6, -0.208772}, {0.7, -0.263182},
  {0.8, -0.00994748}, {0.9, 0.13955}, {1., -0.0861754},
  {1.1, -0.562744}, {1.2, -0.884539}, {1.3, -0.743297},
  {1.4, -0.221009}, {1.5, 0.273882}, {1.6, 0.369945}, {1.7, 0.0631134},
  {1.8, -0.288775}, {1.9, -0.301581}, {2., 0.0781706}}
```

The lhs variables are easily determined when the proportionality constant resultant from a mechanical system based on 5 kg and an electrical one based on 1 H is considered. That would be

1 H, 2 Ω, and $\frac{1}{12}$ Farad

To solve the rhs I can first solve the coefficient situation,

In[59]:=
```
Solve[5 * 10 * x * Cos[10 t] - 220 * Cos[ 10 t] == 0, x]
```

Out[59]= $\{\{x \to \frac{22}{5}\}\}$

and then consider that because what is wanted is the derivative of the electromotive force, I will be looking at

```
-4.4 Sin[10 t]
```

as the rhs. The text answer agrees, except it does not show a negative sign, and in terms of making up the system equation I believe the voltage expression on the rhs is better left unsigned.